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Multiperipheral Dynamics at General Momentum Transfer*

MARCELLO CIAFALONI†
Department of Physics, University of California, Berkeley, California 94720

AND

CARLETON DETAR AND MICHAEL N. MISHELOFF
Lawrence Radiation Laboratory, University of California, Berkeley, California 94720

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We extend the group-theoretical analysis of the multiperipheral integral equation of Chew, Goldberger, and Low to general momentum transfers. Using a set of variables for the multiparticle phase space analogous to those of Bali, Chew, and Pignotti, we obtain, through the $O(2,1)$ symmetry, a partial diagonalization of the equation, without requiring asymptotic approximations to the phase space. As an example, we apply our technique to a multi-Regge model and an Amati-Fubini-Stanghellini-type model.

I. INTRODUCTION

INTEREST in the multiperipheral model of Fubini and collaborators[1] revived when Chew, Goldberger, and Low[2] (CGL) noticed that a generalization of the model provided the framework for a bootstrap program directly involving Regge parameters.[3] They proposed an integral equation[4] which provides a powerful tool for investigating the role of multiparticle unitarity in determining the dynamics of high-energy peripheral processes. The equation has been studied both at zero momentum transfer ($t=0$) and at $t<0$ by several authors,[4–7] who made use of various asymptotic approximations to the phase space in order to achieve a partial diagonalization of the equation. Such an approach is very fruitful since it yields important information about the qualitative features of the model.

It is an empirical fact, however, that the important range of intermediate particle subenergies is not very high. We present here a procedure for exploiting fully the $O(2,1)$ symmetry of the CGL equation with no approximations to the phase space. The burden of more carefully representing the low and intermediate particle subenergies now lies with the choice of the model. Our scheme should provide some insight into the validity of the approximations made in the Mellin-transform approach. In particular, it exhibits some interesting effects of correlations among phase-space variables which may be of consequence even in asymptotic calculations.

The central problem in diagonalizing the CGL equation with an exact treatment of phase space is to find a proper set of kinematical variables.[8] Bali, Chew, and Pignotti[9] (BCP) defined as variables the momentum transfers squared and a set of "angular" variables which are asymptotically proportional to the subenergies. They were, more precisely, the parameters of the three-dimensional Lorentz group which preserve the momentum transfers in the multiperipheral chain (Fig. 1). These variables were adequate for the analysis at $t=0$, where the production amplitude and its complex conjugate in the unitarity integrand may be expressed in terms of the same variables. Making use of factorization at the Regge poles in the multiple $O(2,1)$ decomposition of the unitarity integrand, Chew and DeTar[10] (CD) derived an equation for the absorptive part of the elastic amplitude at $t=0$, which can be partially diagonalized by using its $O(3,1)$ symmetry.[11]

At $t<0$ the amplitude and its complex conjugate are no longer simultaneously evaluated at the same point in phase space, and so we must choose a new set of variables. Consider the unitarity diagram in Fig. 1 with the upper and lower momentum transfers Q_u and Q_l with squares t_u and t_l.[12] In a reference frame in which the over-all momentum transfer Q has only a z component $(-t)^{1/2}$, we have

$$Q_{l,u}=(\mathbf{k},\,w\pm\tfrac{1}{2}(-t)^{1/2}),$$

where both w and the magnitude of the Lorentz three-vector $\mathbf{k}$ are fixed in terms of t_l and t_u. Therefore the subenergy s_i is, for fixed t_l's and t_u's, a function of $\mathbf{k}_{i-1}\cdot\mathbf{k}_{i+1}$ and asymptotically proportional to it.

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[12] Four-vectors are expressed in the form (P_t,P_x,P_y,P_z) with the metric $P\cdot P=P_t^2-P_x^2-P_y^2-P_z^2$.

We are led in a natural way to consider the little groups of the **k**'s instead of those of the Q_l's and Q_u's. Since the most important contribution to the phase space comes from spacelike **k**'s (Sec. II), these little groups are noncompact, one-parameter $O(1,1)$ groups, and these parameters will be our "angular" variables.

In reconstructing the CGL equation we first project the unitarity integrand onto the $O(1,1)$ groups. It is at the poles in the $O(1,1)$ quantum number that we wish to make the factorization assumption which underlies the CGL multiperipheral model. For each Regge pole with factorizable residue in the production amplitude, the $O(1,1)$ partial-wave amplitude will contain an infinite sequence of integrally spaced $O(1,1)$ poles with factorizable residues. For this approach to be useful we assume that, by including only a few leading $O(1,1)$ poles, which are derived from the first few Regge poles, we obtain an adequate average representation of the low-energy region. It is, of course, not necessary that this assumption be made at every link in the multiperipheral chain. We treat a model of the type in Ref. 1 (the "AFS-type model") as an example of a model which does not require such an extreme assumption.[13]

In the present paper we deal essentially with the definition of our variables and the crossed partial-wave analysis of the resulting equation. The precise connection with the BCP expansion will be discussed in a forthcoming paper, together with the $t=0$ limit. Moreover, we do not study here the kinematical singularities of our production amplitudes in the nonleading $O(1,1)$ contributions.

In Sec. II we define our variables and we use them in deriving an exact expression for the many-body phase space, which is suitable for establishing our multiperipheral equation. To illustrate the use of our scheme, we construct the integral equation for both the leading-power multi-Regge model and the AFS-type model in Sec. III. The crossed partial-wave analysis is given in Sec. IV. A remarkable technical result is that the kernel of our partial-wave equation is analytic and well behaved in the right half l plane, since we use a basis in which the relevant representation functions of the $O(2,1)$ group are second-type Legendre functions.

II. KINEMATICS AND PHASE SPACE

The kinematical analysis at $t<0$ proceeds by direct analogy with the approach of BCP and CD. We begin with a review of the key features of their method.

In expressing the multiparticle phase space in terms of group variables, BCP selected a sequence of standard Lorentz frames, corresponding to a given arrangement

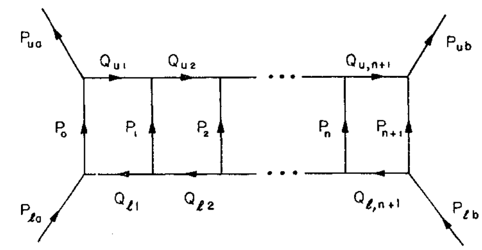

FIG. 1. Momentum-conservation diagram for the $(n+2)$-body contribution to the unitarity sum.

of the outgoing particles in the process

$$la+lb\to 0+1+\cdots+(n+1). \tag{2.1}$$

Associated with each four-momentum transfer $Q_{l,i}$ (see Fig. 1) were a right standard frame (li,r) in which[12]

$$\begin{aligned} Q_{l,i}&=[0,0,0,(-t_{l,i})^{1/2}],\\ Q_{l,i+1}&=(-t_{l,i+1})^{1/2}(\sinh q_{l,i},0,0,\cosh q_{l,i}) \end{aligned}\tag{2.2}$$

and a left standard frame (li,l) in which

$$\begin{aligned} Q_{l,i}&=[0,0,0,(-t_{l,i})^{1/2}],\\ Q_{l,i-1}&=(-t_{l,i-1})^{1/2}(-\sinh q_{l,i-1},0,0,\cosh q_{l,i-1}). \end{aligned}\tag{2.3}$$

The two frames were related by an $O(2,1)$ transformation, $g_{li}=e^{-iJ_z\mu_{li}}e^{-iK_x\zeta_{li}}e^{-iJ_z\nu_{li}}$, which preserved the z axis.[14] In terms of the parameters of g_{li}, the four-vector $Q_{l,i-1}$ assumed, in the frame (li,r), the form

$$\begin{aligned} Q_{l,i-1}=(-t_{l,i-1})^{1/2}(&-\sinh q_{l,i-1}\cosh\zeta_{li},\ \sinh q_{l,i-1}\sinh\zeta_{li}\\ &\times\cos\nu_{li},-\sinh q_{l,i-1}\sinh\zeta_{li}\sin\nu_{li},\ \cosh q_{l,i-1}). \end{aligned}\tag{2.4}$$

From the standpoint of the frame (li,r) this was an adequate parametrization of $Q_{l,i-1}$ under the assumption that $t_{l,i-1}<0$ and $t_{li}<0$. This observation facilitated the change of integration variables. The boost ζ_{li} was connected with the subenergy $s_i\sim -2Q_{li-1}\cdot Q_{li+1}$, thereby providing a framework for the multi-Regge expansion. After linking the frames (li,r) and $(li+1,l)$ with a pure z boost q_{li}, it is possible to go from a particular rest frame of particle lb to a particular rest frame of particle la via all intervening standard frames with the transformation

$$r_{la}q_{l0}g_{l1}q_{l1}\cdots g_{l,n+1}q_{l,n+1}r_{lb}.$$

(The rotations r_{la} and r_{lb} are taken in the rest frames of particles la and lb.)

In constructing a recursive expression for the $(n+2)$-body phase space, CD introduced the Lorentz transformation

$$a_{li}=b_{la}r_{la}q_{l0}g_{l1}q_{l1}\cdots q_{l,i-1}g_{li}, \tag{2.5}$$

which transformed four-momenta from their configura-

[13] Ball and Marchesini have studied the self-consistency of this model using the partial-wave analysis of a Wick-rotated Bethe-Salpeter equation, the absorptive part of which is an integral equation of the type in Ref. 1: J. S. Ball and G. Marchesini, Phys. Rev. **188**, 2209 (1969). Here we give the crossed partial-wave analysis of the unitarity equation at $t<0$ directly.

[14] To define g_{li} unambiguously, it is necessary to fix the initial and final z rotations μ_{li} and ν_{li} by attaching to conditions (2.2) and (2.3) a definition of the y axis. This can be accomplished by specifying a standard form for $Q_{l,i+2}$ and $Q_{l,i-2}$, respectively, in these frames, as did BCP, or by making use of the spin degree of freedom, as did CD.

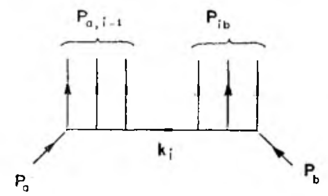

FIG. 2. Lorentz three-momentum diagram corresponding to Fig. 1.

tion in the frame (li,r) to their configuration in a general reference frame. The incomplete absorptive part $B(a_l,t_l)$ which appeared in the integral equation at $t=0$ was a function of a Lorentz transformation of the type a_{li}. The equation was partially diagonalized by projecting $B(a_l,t_l)$ onto representations of the Lorentz group.[11]

At $t<0$ we shall construct an analogous function $B(a,t_l,t_u)$, which depends upon an $O(2,1)$ transformation a. This transformation preserves the over-all four-momentum transfer

$$Q=[0,0,0,(-t)^{1/2}], \tag{2.6}$$

and plays a role analogous to the $O(3,1)$ transformation a_l.

If we fix Q in this way throughout, the components of the four-momentum transfers[15]

$$\begin{aligned} Q_{u,i}&=[\mathbf{k}_i,\ w_i+\tfrac{1}{2}(-t)^{1/2}],\\ Q_{l,i}&=[\mathbf{k}_i,\ w_i-\tfrac{1}{2}(-t)^{1/2}] \end{aligned} \tag{2.7}$$

are partially determined by the constraints

$$Q_{ui}-Q_{li}=Q\,,\quad Q_{ui}^2=t_{ui}\,,\quad Q_{li}^2=t_{li}\,,$$

with the result that

$$\begin{aligned} w_i&=(t_{li}-t_{ui})/2(-t)^{1/2}\,,\\ \mathbf{k}_i\cdot\mathbf{k}_i&=-\lambda(t_{li},t_{ui},t)/4t\,,\\ \lambda(a,b,c)&=a^2+b^2+c^2-2ab-2ac-2bc\,. \end{aligned} \tag{2.8}$$

The key to the analysis at $t<0$ is to recognize that the Lorentz three-vector $\mathbf{k}_i$ plays a role analogous to the Q_{li}. In effect, the z component has been set aside, with the result that the $O(3,1)$ symmetry is reduced to an $O(2,1)$ symmetry. In place of $O(2,1)$, the group preserving the form of Q_{li}, we introduce the $O(1,1)$ or $O(2)$ group, which preserves the form of $\mathbf{k}_i$. As before, large subenergies at *fixed* t_{li}, t_{ui} can occur only when the scalar product $\mathbf{k}_{i-1}\cdot\mathbf{k}_{i+1}$ is large.

Except at the ends of the chain for a fixed value of t, the $\mathbf{k}$'s are spacelike in the sense of three-vectors. This follows from a condition on the invariant three-vector masses analogous to the familiar condition for spacelike four-momentum transfers. Referring to Fig. 2, one sees that if

$$\begin{aligned} \mathbf{P}_{a,i-1}^2&>\mathbf{P}_a^2=\lambda(m_{la}^2,m_{ua}^2,t)/4t\,,\\ \mathbf{P}_{ib}^2&>\mathbf{P}_b^2=\lambda(m_{lb}^2,m_{ub}^2,t)/4t\,, \end{aligned} \tag{2.9}$$

then $\mathbf{k}_i^2<0$. The minimum three-vector mass $\mathbf{P}_i^2$ is m_i^2, the four-vector mass. Hence the constraint (2.9) will automatically be satisfied for a particular value of t after a sufficient number of particle momenta have been included in $\mathbf{P}_{a,i-1}$ and $\mathbf{P}_{i,b}$. For pairwise equal masses ($m_{la}=m_{ua}$ and $m_{lb}=m_{ub}$), $\mathbf{k}_i^2$ is negative when

$$\begin{aligned} s_{a,i-1}+w_i^2&\geq m_a^2-\tfrac{1}{4}t\,,\\ s_{i,b}+w_i^2&\geq m_b^2-\tfrac{1}{4}t\,, \end{aligned} \tag{2.10}$$

where $s=P^2$ is the four-vector mass. The positions of the Regge poles in the elastic absorptive part are determined by the central part of the chain, the ends of the chain serving only to define the pole residues. Hence for notational convenience we shall treat the more important case of spacelike internal $\mathbf{k}_i$ and shall later indicate the simple generalization to timelike $\mathbf{k}_i$, which occur only at the ends of the chain.

We define a sequence of standard frames (i,l) and (i,r) by analogy with (2.2)–(2.4). In frame (i,r),[16]

$$\begin{aligned} \mathbf{k}_i&=(0,k_i,0)\,,\\ \mathbf{k}_{i+1}&=(k_{i+1}\sinh q_i,k_{i+1}\cosh q_i,0)\,, \end{aligned} \tag{2.11}$$

and in frame (i,l),

$$\begin{aligned} \mathbf{k}_i&=(0,k_i,0)\,,\\ \mathbf{k}_{i-1}&=(-k_{i-1}\sinh q_{i-1},k_{i-1}\cosh q_{i-1},0)\,, \end{aligned} \tag{2.12}$$

where $k_i^2=-\mathbf{k}_i\cdot\mathbf{k}_i$.

Because $\mathbf{k}_i$ is along the x axis in both frames, (i,l) and (i,r) are related by an $O(1,1)$ transformation, namely, a y boost ζ_i which preserves at once the x and z axes. Hence in frame (i,r),

$$\begin{aligned} \mathbf{k}_{i-1}=(-k_{i-1}\sinh q_{i-1}\cosh\zeta_i,k_{i-1}\cosh q_{i-1},\\ k_{i-1}\sinh q_{i-1}\sinh\zeta_i)\,. \end{aligned} \tag{2.13}$$

The subenergy $s_i=(P_{i-1}+P_i)^2$ is proportional to $\cosh\zeta_i$ for large ζ_i and fixed t_{li}, t_{ui}:

$$\begin{aligned} s_i&\sim-2\mathbf{k}_{i-1}\cdot\mathbf{k}_{i+1}\\ &\sim 2k_{i-1}k_{i+1}\sinh q_{i-1}\sinh q_i\cosh\zeta_i\,, \end{aligned} \tag{2.14}$$

which follows from (2.11) and (2.13).

We have introduced the x boost q_i to relate the frames (i,r) and $(i+1,\,l)$. From the constraints

$$Q_{l,i+1}-Q_{li}=P_i\,,\quad P_i^2=m_i^2\,,$$

$\cosh q_i$ can be calculated as a function of the momentum transfers or, equivalently, the k's and w's:

$$\cosh q_i=\frac{k_i^2+k_{i+1}^2+(w_i-w_{i+1})^2+m_i^2}{2k_ik_{i+1}}\equiv z_i\,,\qquad (i=1,2,\ldots,n)\,. \tag{2.15}$$

[15] The three-vector $\mathbf{k}$ always refers to the components (Q_t,Q_x,Q_y).

[16] This specification defines the frames (i,r) and (i,l) up to a reflection in the xz plane. There is no rotational freedom left as in the BCP frames (Ref. 14).

If $\mathbf{k}_i \cdot \mathbf{k}_i$ in Eq. (2.11) had been positive, we would have written

$$\mathbf{k}_i = (E_i, 0, 0), \tag{2.16}$$

where $E_i^2 = \mathbf{k}_i \cdot \mathbf{k}_i$. In this case the y boost ζ_i must be replaced by a z rotation ϕ_i [an $O(2)$ transformation] as the transformation relating the frames (i,l) and (i,r) and preserving the form of $\mathbf{k}_i$. The form (2.16) is required at the very ends of the chain. Here we define the frame (a,r) in which

$$\begin{aligned} \mathbf{P}_a &= (E_a, 0, 0), \\ \mathbf{k}_1 &= (k_1 \sinh q_0, k_1 \cosh q_0, 0), \end{aligned} \tag{2.17}$$

the frame (b,l) in which

$$\begin{aligned} \mathbf{P}_b &= (E_b, 0, 0), \\ \mathbf{k}_{n+1} &= (-k_{n+1} \sinh q_{n+1}, k_{n+1} \cosh q_{n+1}, 0), \end{aligned} \tag{2.18}$$

and the frame (b,r) where

$$\begin{aligned} \mathbf{P}_b &= (E_b, 0, 0), \\ \mathbf{k}_{n+1} &= (-k_{n+1} \sinh q_{n+1}, k_{n+1} \cosh q_{n+1} \cos\phi_b, \\ &\qquad -k_{n+1} \cosh q_{n+1} \sin\phi_b). \end{aligned} \tag{2.19}$$

Corresponding to Eq. (2.15), we find

$$\begin{aligned} \sinh q_0 &= \frac{m_0^2 - E_a^2 + k_1^2 + (w_a + w_1)^2}{2E_a k_1} \equiv z_0, \\ \sinh q_{n+1} &= \frac{m_{n+1}^2 - E_b^2 + k_{n+1}^2 + (w_b - w_{n+1})^2}{2E_b k_{n+1}} \equiv z_{n+1}. \end{aligned} \tag{2.20}$$

From these results the procedure for generalizing to an arbitrary choice of spacelike and timelike three-momentum transfers should be obvious.

For vertices with adjacent spacelike $\mathbf{k}_i$ on both sides, it is evident from (2.15) that $\cosh q_i \geqslant 1$, and from (2.11) we see that

$$q_i \geqslant 0 \tag{2.21}$$

if P_i is to be forward timelike. From (2.19) it is evident that for timelike-spacelike vertices, q may be negative.

Pursuing our analogy further, we define the $O(2,1)$ transformation[17]

$$a_i = b_a \phi_a q_0 \zeta_1 q_1 \cdots q_{i-1} \zeta_i, \tag{2.22}$$

where b_a is an arbitrary $O(2,1)$ transformation which preserves Q. The construction of the $(n+2)$-body phase space in terms of the $O(1,1)$ and $O(2)$ group variables ζ_i, ϕ_b and the variables k_i, w_i proceeds in much the same way as before. The familiar expression for the phase space in terms of the four-momenta,

$$\begin{aligned} &d\Phi_{n+2}(P_{la}, P_{lb}) \\ &= d^4P_0 \delta^{(+)}(P_0^2 - m_0^2) d^4P_1 \delta^{(+)}(P_1^2 - m_1^2) \cdots \\ &\quad d^4P_{n+1} \delta^{(+)}(P_{n+1}^2 - m_{n+1}^2) \delta^4\Big(\sum_i P_i - P_{la} - P_{lb}\Big), \end{aligned} \tag{2.23}$$

may be rewritten in terms of the components of the four-momentum transfers $Q_{li} = [\mathbf{k}_i, w_i - \frac{1}{2}(-t)^{1/2}]$:

$$\begin{aligned} d\Phi_{n+2}(P_{la}, P_{lb}, t) &= \delta^{(+)}(P_0^2 - m_0^2) d^3k_1 dw_1 \\ &\times \delta^{(+)}(P_1^2 - m_1^2) \cdots d^3k_{n+1} dw_{n+1} \delta^{(+)}(P_{n+1}^2 - m_{n+1}^2). \end{aligned} \tag{2.24}$$

We picture the phase-space volume element as being defined for a fixed initial $O(2,1)$ transformation b_a, which defines $\mathbf{P}_a$, and a fixed over-all $O(2,1)$ transformation b_b, which defines $\mathbf{P}_b$:

$$b_b = b_a (\phi_a q_0 \zeta_1) q_1 \cdots \zeta_{n+1} q_{n+1} \phi_b. \tag{2.25}$$

If we integrate first over $d^3k_1 dw_1$, next over $d^3k_2 dw_2$, and so on, from the standpoint of the first integration a_2 is a constant Lorentz transformation, since $a_2 = b_b \phi_b^{-1} q_{n+1}^{-1} \cdots \zeta_3^{-1} q_2^{-1}$ does not depend upon $\mathbf{k}_1$ and w_1. Transforming $\mathbf{k}_1$ by a_2^{-1} brings $\mathbf{k}_1$ to its configuration in the frame $(2,r)$ where the parametrization (2.13) applies. We make use of this parametrization to change variables:

$$d^3k_1 \to k_1^2 dk_1 d\cosh q_1 d\zeta_2.$$

Proceeding to the $d^3k_2 dw_2$ integration, we regard a_3 as being fixed by the subsequent integration variables. Repeating this argument, we make the replacement

$$d^3k_i \to k_i^2 dk_i d\cosh q_i d\zeta_{i+1} \tag{2.26}$$

for $i = 1, 2, \ldots, n$. Finally, regarding b_b as fixed, we make use of Eq. (2.19) to replace d^3k_{n+1}:

$$d^3k_{n+1} \to k_{n+1}^2 dk_{n+1} d\sinh q_{n+1} d\phi_b. \tag{2.27}$$

The mass-shell constraint on P_i^2 may be used to eliminate the integration over q_i:

$$\begin{aligned} \delta^{(+)}(P_i^2 - m_i^2) d\cosh q_i &= 1/2k_i k_{i+1} \\ &\qquad (\text{for } i = 1, 2, \ldots, n), \\ \delta^{(+)}(P_{n+1}^2 - m_{n+1}^2) d\sinh q_{n+1} &= 1/2E_b k_{n+1}, \\ \delta^{(+)}(P_0^2 - m_0^2) &= (1/2E_a k_1) \delta(\sinh q_0 - z_0). \end{aligned} \tag{2.28}$$

Putting together (2.24) and (2.26)–(2.28), we write, finally,

$$\begin{aligned} d\Phi_{n+2}(b_a, b_b, t) &= (1/2^{n+2} E_a E_b) dk_1 dw_1 d\zeta_2 dk_2 dw_2 d\zeta_3 \cdots \\ &\quad dk_{n+1} dw_{n+1} d\phi_b \delta(\sinh q_0 - z_0). \end{aligned} \tag{2.29}$$

The range of variables $0 \leqslant k_i \leqslant \infty$ and $-\infty \leqslant w_i \leqslant \infty$ spans that portion of the phase space in which $\mathbf{k}_i$ is spacelike. The complete phase space must, of course, include an integration over $dE_i dw_i d\phi_{i+1}$ for timelike $\mathbf{k}_i$, where $0 \leqslant E_i \leqslant \infty$, $-\infty \leqslant w_i \leqslant \infty$. An additional constraint upon the range of integration is imposed by the

[17] We use the same symbol for the Lorentz transformations ϕ, q, and ζ as their parameters.

δ function in (2.29), since q_0 depends upon all the integration variables through (2.25). This constraint places an upper bound on the E_i which is eventually reduced to zero after a finite distance along the chain.

The recursive property of the phase space may be stated as follows:

$$d\Phi_{n+2}(b_a,b_b,t)=d\Phi_{n+1}'(b_a,a_{n+1},t)\frac{dk_{n+1}}{2E_b}dw_{n+1}d\phi_b\,,$$

$$d\Phi_{i+1}'(b_a,a_{i+1},t)=d\Phi_i'(b_a,a_i,t)\tfrac{1}{2}dk_idw_id\zeta_{i+1} \quad (\text{for } i=1,2,\ldots,n)\,, \tag{2.30}$$

$$d\Phi_1'(b_a,a_1,t)=\delta^{(+)}(\sinh q_0-z_0)/2E_a\,,$$

with the proviso that

$$b_b=a_{n+1}q_{n+1}\phi_b\,,$$
$$a_{i+1}=a_iq_i\zeta_{i+1}\,, \tag{2.31}$$
$$a_1=b_a\phi_aq_0\zeta_1\,.$$

It may be helpful to remark that when the δ-function constraint is satisfied in the integration $d\Phi_i'$, it is automatically satisfied in $d\Phi_{i+1}'$ because of the second condition (2.31), which is consistent with (2.22) and (2.25).

Because there was no rotational freedom left in defining our standard frames in (2.11)–(2.13), we cannot use the simple device of replacing a helicity sum with an integration over a rotation in the little group of $\mathbf{k}_i$ as CD did with the little groups of Q_{li}. The sum over spin degrees of freedom must therefore be performed explicitly. The correct procedure using the BCP amplitudes will be described in a forthcoming paper. Here, for the sake of simplicity, we shall treat only pions in the intermediate states.

III. FORM OF AMPLITUDE AND CONSTRUCTION OF MULTIPERIPHERAL INTEGRAL EQUATION

A. Multi-Regge Model

In order to construct the multiperipheral integral equation for $t<0$, we must first express $M_l^{(n+2)}$ and $M_u^{(n+2)}$, the amplitudes for the processes $la+lb\to 0+1+\cdots+(n+1)$ and $ua+ub\to 0+1+\cdots+(n+1)$, respectively, in terms of our variables. The expressions are similar because our choice of variables is symmetrical with respect to the upper and lower amplitudes. We therefore drop the labels l and u for the moment.

If $M^{(n+2)}$ is a square-integrable function of the ζ_i's, it can be written in terms of its projection onto the unitary irreducible representations of the appropriate groups:

$$M^{(n+2)}(\phi_a,\zeta_1,\ldots,\zeta_{n+1},\phi_b;\,E_a,w_a,k_1,w_1,\ldots,k_{n+1},w_{n+1},E_b,w_b;\,t)=(2\pi)^{-\frac{1}{2}(n+3)}\sum_{m_am_b}(-i)^{n+1}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{i\infty}d\mu_1\cdots d\mu_{n+1}e^{im_a\phi_a}$$
$$\times e^{\mu_1\zeta_1}\cdots e^{\mu_{n+1}\zeta_{n+1}}e^{im_b\phi_b}\tilde{M}^{(n+2)}(m_a,\mu_1,\ldots,\mu_{n+1},m_b;\,E_a,\ldots,w_b;\,t)\,. \tag{3.1}$$

For non-square-integrable functions of physical interest, Eq. (3.1) is valid provided that the contour of integration is deformed away from the imaginary axis in an appropriate way.

For example, if we assume that $\tilde{M}^{(n+2)}$ is a meromorphic function of the μ_i's, poles at $\mu_i=\pm\alpha_i$ give a contribution to the amplitude of the form

$$M^{(n+2)}(\phi_a,\zeta_1,\ldots,\zeta_{n+1},\phi_b;\,E_a,\ldots,w_b;\,t)=\text{const}\sum_{m_am_b}e^{im_a\phi_a}e^{im_b\phi_b}e^{\alpha_1|\zeta_1|}\cdots e^{\alpha_{n+1}|\zeta_{n+1}|}R(E_a,w_a,k_1,w_1,\ldots,E_b,w_b;\,t)\,, \tag{3.2}$$

where the μ_i contour has been moved either left or right, depending upon the sign of ζ_i.

In order to obtain a physically meaningful form for $M^{(n+2)}$, let us evaluate the multi-Regge amplitude in terms of our variables, keeping only leading-order terms. The asymptotic form of the amplitude is given by

$$M^{(n+2)}\sim\sum_{\gamma_i}\tilde{\beta}^{\gamma_1}(t_1)s_1^{\alpha_{\gamma 1}(t_1)}\tilde{\beta}^{\gamma_1,\gamma_2}(t_1,t_2,\omega_1)s_2^{\alpha_{\gamma 2}(t_2)}\cdots s_n^{\alpha_{\gamma n}(t_n)}\tilde{\beta}^{\gamma_n,\gamma_{n+1}}(t_n,t_{n+1},\omega_n)s_{n+1}^{\alpha_{\gamma n+1}(t_{n+1})}\tilde{\beta}^{\gamma_{n+1}}(t_{n+1})\,, \tag{3.3}$$

where ω_i is the Toller angle.[9] We have written $\alpha_{\gamma i}$ in the superscripts instead of α_{γ_i} to avoid typesetting problems.

We must evaluate s_i and ω_i in terms of our variables. Recall that the asymptotic form of the subenergy (2.14) is given by

$$s_i\sim k_{i-1}k_{i+1}\sinh q_{i-1}\sinh q_ie^{|\zeta_i|}\,. \tag{3.4}$$

As for ω_i, we have[18]

$$\omega_i\to\omega_i(k_i,w_i,k_{i+1},w_{i+1},m_i^2,t,\text{sgn}\zeta_i,\text{sgn}\zeta_{i+1})\,, \tag{3.5}$$

as $\zeta_i,\zeta_{i+1}\to\infty$. Because the extra variables $\text{sgn}\zeta_i$ are needed to label the residues at the $O(1,1)$ poles, in the following we discard the ω_i dependence (see Acknowledgments).

[18] In terms of the variables for M_l,

$$(\cos\omega_{li}+\cosh q_{li})\to(t_{l,i}t_{l,i+1})^{1/2}(\sinh q_{li})^2(\text{sgn}\zeta_i\,\text{sgn}\zeta_{i+1}+\cosh q_i)/k_ik_{i+1}(\sinh q_i)^2.$$

Substituting Eqs. (3.4) and (3.5) into Eq. (3.3), we easily find that

$$M^{(n+2)}\sim \sum_{m_a m_b \gamma_i} e^{im_a\phi_a}\beta_{m_a}{}^{\gamma_1}(E_a,w_a,k_1,w_1,m_0{}^2,t)e^{\alpha_{\gamma 1}(t_1)|\zeta_1|}\beta^{\gamma_1\gamma_2}(k_1,w_1,k_2,w_2,m_1{}^2,t)e^{\alpha_{\gamma 2}(t_2)|\zeta_2|}\cdots e^{\alpha_{\gamma n}(t_n)|\zeta_n|}$$
$$\times\beta^{\gamma_n\gamma_{n+1}}(k_n,w_n,k_{n+1},w_{n+1},m_n{}^2,t)e^{\alpha_{\gamma n+1}(t_{n+1})|\zeta_{n+1}|}\beta_{m_b}{}^{\gamma_{n+1}}(k_{n+1},w_{n+1},E_b,w_b,m_{n+1}{}^2,t)e^{im_b\phi_b}, \quad (3.6)$$

where the kinematic factors k_i and $\sinh q_i$ have been absorbed into the residue functions.

Thus a Regge pole at $\alpha_{\gamma i}(t_i)$ in $M^{(n+2)}$ generates, in leading order, $O(1,1)$ poles at $\mu_i=\pm\alpha_{\gamma_i}(t_i)$. In general, we expect a Regge pole to generate a sequence of $O(1,1)$ poles spaced by integers. The residues at the poles are factorizable, enabling us to derive an integral equation for the absorptive part of the amplitude. We note that whereas the $O(1,1)$ vertex functions depend upon the over-all momentum transfer,[19] the positions of the $O(1,1)$ poles, considered as a function of t_l and t_u, are independent of it.

We are now in a position to derive an integral equation for determining $A(b_a{}^{-1}b_b,t)$, the absorptive part of the amplitude ($la, lb\to ua, ub$). As noted in Sec. II, timelike $\mathbf{k}_i$ occur only at the ends of the chain, and so do not affect the position of the output Regge poles. For the sake of convenience, therefore, we write the integral equation integrating only over spacelike $\mathbf{k}_i$.[20] We assume that $M_l{}^{(n+2)}$ and $M_u{}^{(n+2)}$ can be approximated by sums of $O(1,1)$ poles with factorizable residues as in (3.6). Restoring the labels l and u, we define m_a, m_b, $\alpha_{\gamma i}$, and $R^{\gamma_i\gamma_{i+1}}$ by

$$m_a=m_{la}-m_{ua}, \quad m_b=m_{lb}-m_{ub},$$
$$\alpha_{\gamma_i}(k_i,w_i,t)=\alpha_{\gamma_{li}}(t_{li})+\alpha_{\gamma_{ui}}(t_{ui}),$$
$$R^{\gamma_i\gamma_{i+1}}(k_i,w_i,k_{i+1},w_{i+1},m_i{}^2,t)$$
$$=\beta_l{}^{\gamma_{li}\gamma_{li+1}}(k_i,w_i,k_{i+1},w_{i+1},m_i{}^2,t)$$
$$\times[\beta_u{}^{\gamma_{ui}\gamma_{ui+1}}(k_i,w_i,k_{i+1},w_{i+1},m_i{}^2,t)]^*. \quad (3.7)$$

The derivation of the integral equation closely parallels that of CD. We merely quote the results. The incomplete absorptive part is the solution of the equation

$$B^{\gamma'}(a';k',w',t)={}_{(0)}B^{\gamma'}(a';k',w';t)$$
$$+\tfrac{1}{2}\sum_\gamma\int dkdwd\zeta' B^\gamma(a;k,w,t)R^{\gamma\gamma'}(k,w,k',w',m^2,t)$$
$$\times e^{\alpha_{\gamma'}(k',w',t)|\zeta'|}, \quad (3.8)$$

where

$$a'=aq\zeta' \quad (3.9)$$

and

$$\cosh q=(1/2kk')[k^2+k'^2+(w-w')^2+m^2]. \quad (3.10)$$

The inhomogeneous term is given by

$${}_{(0)}B^{\gamma'}(a';k',w';t)=\sum_{m_{la}m_{ua}}\frac{1}{2E_a}\delta(\sinh q_0-z_0)e^{im_a\phi_a}$$
$$\times R_{m_{la}m_{ua}}{}^{\gamma'}(E_a,w_a,k',w',m_0{}^2,t)e^{\alpha_{\gamma'}(k',w',t)|\zeta'|}, \quad (3.11)$$

with $a'=\phi_a q_0\zeta'$. The complete absorptive part $A(b_a{}^{-1}b_b,t)$ is determined from B^γ by

$$A(b_a{}^{-1}b_b,t)=\frac{1}{2E_b}\sum_{m_{lb}m_{ub}\gamma}\int dkdwd\phi_b B^\gamma(b_a{}^{-1}a;k,w;t)$$
$$\times R_{m_{lb}m_{ub}}{}^\gamma(k,w,E_b,w_b,m^2,t)e^{im_b\phi_b}, \quad (3.12)$$

with

$$b_b=aq\phi_b. \quad (3.13)$$

B. AFS-Type Model

In the model of Fubini and collaborators[1] the factorization assumption in the production amplitudes is introduced through the pion-pole dominance, and the building blocks of the multiperipheral chain are the (off-shell) pion-pion scattering amplitudes.

In evaluating the unitarity integral (Fig. 3) we can make, on the momentum transfers Q_l's and Q_u's, the same change of variables as in Sec. II, while the remaining loop integrals simply give the off-shell elastic π-π cross section A_2 for each link of the chain. So we have

$$A_{n+2}(b_a{}^{-1}b_b,t)$$
$$=\int A_2(\sinh q_0;E_a,w_a;k_1,w_1)\prod_{i=1}^n k_i{}^2dk_idw_i$$
$$\times d\cosh q_i d\zeta_{i+1}G_iA_{2i}dk_{n+1}dw_{n+1}d\sinh q_{n+1}d\phi_b$$
$$\times k_{n+1}{}^2G_{n+1}A_2(\sinh q_{n+1};k_{n+1},w_{n+1};E_b,w_b), \quad (3.14)$$

where

$$G_i\equiv(t_{li}-\mu^2)^{-1}(t_{ui}-\mu^2)^{-1},$$
$$A_{2i}\equiv A_2(\cosh q_i;k_i,w_i;k_{i+1},w_{i+1}), \quad (3.15)$$
$$\cosh q_i=[k_i{}^2+k_{i+1}{}^2+(w_i-w_{i+1})^2+s_i]/2k_ik_{i+1},$$
$$s_i\geq 4\mu^2$$

and $\sinh q_0$ is defined in a similar way.

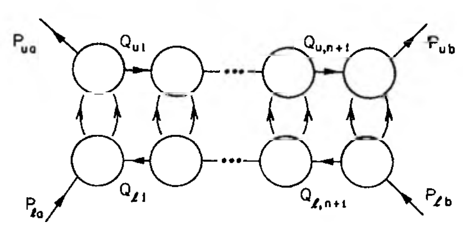

FIG. 3. Unitarity contribution for the AFS-type model.

[19] The $O(1,1)$ variables are defined in reference frames which are partly determined by Q, Eq. (2.6). So this t dependence is not surprising. The $O(1,1)$ expansion is natural for the unitarity integrand, but not quite for the production amplitudes themselves.

[20] It is always possible to recast an integral equation of the type (3.8) in terms of $\bar{B}=B-B_n$, where B_n represents the sum of the first n terms in B, obtained by iterating the original equation. Since the timelike $\mathbf{k}$'s disappear after a finite number of iterations, one can always obtain, with this device, an integral equation involving strictly spacelike $\mathbf{k}$'s.

The analogy of Eq. (3.14) with the multi-Regge model is apparent. The $\delta(\cosh q_i - z_i)$ in the phase space is now replaced by $A_2(\cosh q_i)\theta(q_i - q_{i,\min})$ and the ζ dependence of Eq. (3.6) has now disappeared, because the exchanged pions are not Reggeized. With the usual procedure[1,10] we get an equation for the incomplete absorptive part:

$$B(a',t) = {}_{(0)}B(a',t) + \int k^2 dk dw d \cosh q d\zeta' B(a'\zeta'^{-1}q^{-1},t) \times G(k,w) A_2(\cosh q; k,w; k',w') , \quad (3.16)$$

where, if a is parametrized by

$$a = e^{-iJ_z\phi} e^{-iK_x\eta} e^{-iK_y\xi} \quad (3.17)$$

and s is the energy, then

$$\begin{aligned} {}_{(0)}B(a,t) &= A_2(\sinh\eta; E_a,w_a; k,w) , \\ \sinh\eta &= [k^2 - E_a^2 + (w-w_a)^2 + s]/2kE_a . \end{aligned} \quad (3.18)$$

The complete absorptive part is obtained from B by the formula

$$A(a',t) = \int k^2 dk dw d \sinh q d\phi\, B(a'\phi^{-1}q^{-1},t) \times G(k,w) A_2(\sinh q; k,w; E_b,w_b) . \quad (3.19)$$

IV. CROSSED PARTIAL-WAVE ANALYSIS

Equations (3.8) and (3.16) have $O(2,1)$ symmetry because both kernels are invariant under the transformation $a' \to ca'$, $a \to ca$, where c is an arbitrary $O(2,1)$ transformation not affecting b_a. To exploit this symmetry, we shall expand $B(a)$ (we drop the k, w variables for the moment) in terms of representation functions of $O(2,1)$.[21] Because of the parametrization of a [Eqs. (2.22) and (3.17)] we shall use a mixed basis, namely, an $O(2)$ basis associated with timelike $\mathbf{k}_a$ and an $O(1,1)$ basis[22,23] associated with spacelike $\mathbf{k}$, where the y-boost generator K_y is diagonal and has eigenvalue ρ ($-\infty < \rho < +\infty$). The representation functions carry an extra index $r = \pm$ because each eigenvalue ρ of K_y occurs twice in the completeness relation. The properties of these representation functions are given in Appendix A, which relies heavily upon the work of Mukunda.[22]

We expand

$$B(a) = \int_C d[l](-i) \int_{-i\infty}^{+i\infty} d\mu \sum_r B_{\mu r}{}^l D_{0,\mu r}{}^l(a) , \quad (4.1)$$

where $\mu \equiv i\rho$, and we assume for simplicity that the helicity difference $m_a = m_{la} - m_{ua} = 0$; C is an infinite contour along $\mathrm{Re}\, l = -\frac{1}{2}$,[24] and

$$d[l] = (8\pi i)^{-1}(2l+1)\cot\pi l\, dl . \quad (4.2)$$

The form of our equations is

$$B(a') = {}_{(0)}B(a') + \int d \cosh q d\zeta' B(a'\zeta'^{-1}q^{-1}) \times K(\cosh q,\zeta') , \quad (4.3)$$

where

$$K(\cosh q,\zeta') = \delta(\cosh q - z) Re^{\alpha|\zeta'|} \quad \text{(multi-Regge)} \quad (4.4a)$$

$$= GA_2(\cosh q) \quad \text{(AFS-type model)} , \quad (4.4b)$$

and all the irrelevant labels have been dropped for simplicity. Substituting (4.1) into (4.3) and making use of the identity

$$D_{0,\mu r}{}^l(a'\zeta'^{-1}q^{-1}) = \sum_{r'} \int_{-i\infty}^{+i\infty} (-i) d\mu' \times D_{0,\mu' r'}{}^l(a') e^{\mu'\zeta'} d_{\mu' r',\mu r}{}^l(q^{-1}) , \quad (4.5)$$

we obtain the partially diagonalized equation

$$B_{\mu' r'}{}^l = {}_{(0)}B_{\mu' r'}{}^l + \sum_r \int_{-i\infty}^{+i\infty} (-i) d\mu B_{\mu r}{}^l K_{\mu r,\mu' r'}{}^l , \quad (4.6)$$

where

$$K_{\mu r,\mu' r'}{}^l = \int d \cosh q d\zeta' K(\cosh q,\zeta') e^{\mu'\zeta'} d_{\mu' r',\mu r}{}^l(q^{-1}) ,$$

$$d_{\mu' r',\mu r}{}^l(q^{-1}) \equiv \langle l,\rho' r' | e^{iK_x q} | l,\rho r \rangle . \quad (4.7)$$

With our r basis (Appendix A), we have

$$d_{\mu'+,\mu-}{}^l(q^{-1}) = d_{\mu'-,\mu+}{}^l(q) = 0 \quad (q>0) , \quad (4.8)$$

provided that the internal vertex boost $q>0$. Because we require positive energies for the outgoing particles, this is always true for vertices with adjacent spacelike $\mathbf{k}$'s [Eq. (2.21)]. Therefore we have, symbolically,

$$B_+{}^l = {}_{(0)}B_+{}^l + B_+{}^l K_{++}{}^l , \quad (4.9a)$$

$$B_-{}^l = {}_{(0)}B_-{}^l + B_+{}^l K_{+-}{}^l + B_-{}^l K_{--}{}^l . \quad (4.9b)$$

Note that the $+$ amplitude is decoupled and can be determined separately. Since, as shown in Appendix B, the output Regge poles are given by the kernel $K_{++}{}^l$ only, from now on we shall concentrate on that equation. The relation between $(-)$ and $(+)$ amplitudes will also be discussed in Appendix B. Note that the

[21] M. Toller, Nuovo Cimento **37**, 631 (1965).
[22] N. Mukunda, J. Math. Phys. **8**, 2210 (1967); Tata Institute of Fundamental Research, Bombay, Report, 1968 (unpublished).
[23] J. K. Kuriyan, N. Mukunda, and E. C. G. Sudarshan, J. Math. Phys. **9**, 2100 (1968).

[24] We assume that all Regge poles are to the left of $\mathrm{Re}\, l = -\frac{1}{2}$, so that this expansion converges properly. When the poles move to the right, the contours have to be distorted accordingly.

representation function occurring in $K_{++}{}^l$ is (Appendix A)

$$d_{\mu'+,\mu+}{}^l(q^{-1})\equiv d_{\mu'\mu}{}^l$$
$$=\frac{1}{2\pi}(\cosh\tfrac{1}{2}q)^{-2l-2}\int_{-\infty}^{+\infty}d\zeta\, e^{\zeta(l+1-\mu')}$$
$$\times(e^\zeta+\tanh\tfrac{1}{2}q)^{-l-1+\mu}(1+e^\zeta\tanh\tfrac{1}{2}q)^{-l-1-\mu}$$
$$=\frac{1}{2\pi}(\sinh\tfrac{1}{2}q)^{-2l-2}(\tanh\tfrac{1}{2}q)^{-(\mu+\mu')}$$
$$\times\frac{\Gamma(l+1+\mu')\Gamma(l+1-\mu')}{\Gamma(2l+2)}$$
$$\times F(l+1+\mu,l+1+\mu';2l+2;-(\sinh\tfrac{1}{2}q)^{-2}),\quad(4.10)$$

and is therefore a pure Q_l-type function. In particular,

$$d_{00}{}^l(q^{-1})=(1/\pi)Q_l(\cosh q).\quad(4.11)$$

Equation (4.9a) is still an integral equation in $\rho=-i\mu$, as is expected in general, ρ being the analog of the intermediate helicity in a t-channel two-body unitarity sum. Considerable simplification is, however, achieved for the kernels (4.4) which represent only the leading $O(1,1)$ poles at each link in the multiparticle amplitude.

In the AFS-type model [Eq. (4.4b)] the kernels $K_{rr'}{}^l$ contain a $\delta(\rho')$ factor, due to the lack of ζ dependence (spinless particles). By factoring the δ function out and restoring the k, w variables, we easily obtain

$$B_+{}^l(k',w')={}_{(0)}B_+{}^l(k',w')+\int k^2dkdw$$
$$\times B_+{}^l(k,w)K^l(k,w;k',w'),\quad(4.12)$$
$$K^l=G(k,w)2\int_{z_{\min}}^{\infty}dz\,A_2(z;k,w;k'w')Q_l(z),$$

where $z_{\min}>1$ is the threshold value of $\cosh q$ in (3.15) with $s=4\mu^2$. Note that K^l is the same partial-wave kernel as the one obtained from the Bethe-Salpeter equation corresponding[25] to the unitarity equation (3.16). This kernel can be obtained either by means of a Wick rotation[13,26] or through the crossed partial-wave analysis.[27]

In the multi-Regge model we can approximate the integral equation in μ with a system of equations coupling the $O(1,1)$ poles together. From Eqs. (4.4a) and (4.7), we have

$$K_{\mu+,\mu'+}{}^l=\frac{2\alpha}{\mu'^2-\alpha^2}Rd_{\mu'\mu}{}^l(q^{-1}),\quad(4.13)$$

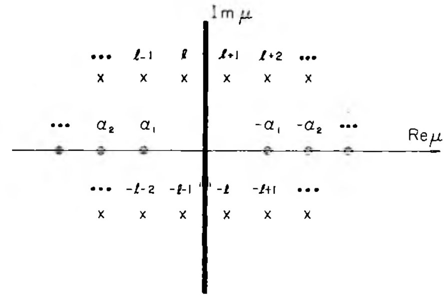

FIG. 4. Poles in the μ plane for the integration of Eq. (4.15).

the modification for more than one $O(1,1)$ pole being obvious.

Because of the analyticity properties in μ, μ' of $d_{\mu'\mu}{}^l$ given in Eq. (4.10), it is evident that $B_{\mu+}{}^l$ has both some "kinematical" poles which can be factored out,

$$B_{\mu+}{}^l=\frac{\Gamma(l+1+\mu)\Gamma(l+1-\mu)}{\Gamma(2l+2)}\hat{B}_\mu{}^l,\quad(4.14)$$

and "dynamical" poles at $\mu=\pm\alpha$. The meaning of the kinematical poles can be seen from the partial-wave projection of Eq. (3.12),

$$A^l=(2E_b)^{-1}\int dkdw(-i)\int d\mu\sum_r B_{\mu r}{}^l d_{0,\mu r}{}^l(q^{-1})$$
$$\text{for } m_b=m_{lb}-m_{ub}=0.\quad(4.15)$$

The pinching of the poles $\mu=\alpha$ and $\mu=l+1+n$ (Fig. 4) ($n=0, 1, \ldots$) gives rise to a singularity in the l plane at $l=\alpha-1-n$, moving with k and w, and therefore to a Regge cut in Eq. (4.15).[28]

By dividing the ζ integration of Eq. (4.10) into the pieces $(-\infty,0)$ and $(0,+\infty)$, we can write

$$\Gamma(l+1+\mu')^{-1}\Gamma(l+1-\mu')^{-1}d_{\mu'\mu}{}^l\Gamma(l+1+\mu)\Gamma(l+1-\mu)$$
$$=d_{\mu\mu'}{}^l=\hat{d}_{\mu\mu'}{}^l+\hat{d}_{-\mu,-\mu'}{}^l,\quad(4.16)$$

where $\hat{d}_{\mu\mu'}{}^l$ has only the poles $\mu=l+1+n$ $(n=0,1,\ldots)$ in the right-half μ plane, and is well behaved when $\mathrm{Re}\mu\to-\infty$. In terms of $\hat{B}_\mu{}^l$, our equation reads

$$\hat{B}_{\mu'}{}^l-{}_{(0)}\hat{B}_{\mu'}{}^l=(-i)\int_{-i\infty}^{+i\infty}d\mu\frac{2\alpha}{\mu'^2-\alpha^2}$$
$$\times\hat{B}_\mu{}^lR(\hat{d}_{\mu,\mu'}{}^l+\hat{d}_{-\mu,-\mu'}{}^l).\quad(4.17)$$

We now displace the μ integration towards the left in

[25] In Ref. 1 the construction is given of a Bethe-Salpeter equation whose absorptive part, due to the Cutkosky rules, is the unitarity equation (3.16). If a Regge-pole expansion of the off-shell π-π amplitude is assumed, such an equation does possess the AFS cuts (Ref. 13).

[26] B. W. Lee and R. Sawyer, Phys. Rev. **127**, 2266 (1962).

[27] L. Sertorio and M. Toller, Nuovo Cimento **33**, 413 (1964).

[28] As shown in Fig. 4, there are also poles at $\mu=\pm(l-n')$, coming from $d_{0,\mu r}{}^l(q^{-1})$, which appears in (4.15) and not in (4.17). The only effect of the additional pinchings is to generate a symmetric cut at $l=-\alpha+n$ in A^l, as expected. This is most easily seen by performing on $B_\mu{}^l$ and $d_{0,\mu r}{}^l$ decompositions similar to (4.16). In this respect the μ-plane singularities here are similar to the l-plane singularities of Toller amplitudes (symmetric under $l\leftrightarrow -l-1$) and a separation of left-hand and right-hand poles simplifies the distortion of the contours.

the μ plane for $\hat{d}_{\mu,\mu'}{}^l$ and towards the right for $\hat{d}_{-\mu,-\mu'}{}^l$ picking up the dynamical poles at $\mu=\pm\alpha$. If we neglect the remaining background integral, we get, finally,

$$b_{\gamma'}{}^l - {}_{(0)}b_{\gamma'}{}^l = 2\pi \sum_{\gamma} b_\gamma{}^l R^{\gamma\gamma'}(\hat{d}_{\alpha_\gamma,\alpha_{\gamma'}}{}^l + \hat{d}_{\alpha_\gamma,-\alpha_{\gamma'}}{}^l)\,, \tag{4.18}$$

where $b_\gamma{}^l$ is the residue of $\hat{B}_\mu{}^l$ at the pole $\mu=\alpha_\gamma$ and we have generalized to the case of several $O(1,1)$ poles.

The background integral represents the contribution of lower-ranking singularities in the input $O(1,1)$ series. Neglecting this integral involves an assumption about the convergence of our solution as we include successively more input singularities. For our method to be useful, the locations and residues of the leading singularities in the l plane of the solution should be determined to a good approximation by a small number of leading singularities in the μ plane. Note that the background integral has its first l-plane singularity at $l=-M-1$ on the left, where $\mu=-M$ is the position of the next singularity in $\hat{B}_\mu{}^l$, which has been neglected. This lends credence to the above-stated assumption.

If we now restore the k, w variables, the approximate Eq. (4.18) reads (γ is short for γ_u, γ_l)

$$\begin{aligned} b_{\gamma'}{}^l(k',w') - {}_{(0)}b_{\gamma'}{}^l(k',w') = \pi \sum_\gamma \int dk dw\, b_\gamma{}^l(k,w) \\ \times R^{\gamma\gamma'}(k,w;k',w')[\hat{d}_{\alpha_\gamma,\alpha_{\gamma'}}{}^l(q^{-1}) + \hat{d}_{\alpha_\gamma,-\alpha_{\gamma'}}{}^l(q^{-1})]\,, \end{aligned} \tag{4.19}$$

where[29]

$$\begin{aligned} {}_{(0)}b_\gamma{}^l(k,w) &= \tilde{\beta}^{\gamma l}(t_l)[\tilde{\beta}^{\gamma u}(t_u)]^*(E_a \cosh q_0)^{\alpha_\gamma} \\ &\quad\times \frac{\Gamma(2l+2)}{\Gamma(l+1+\alpha_\gamma)} Q_l{}^{\alpha_\gamma}(z_0)\,, \\ \hat{d}_{\alpha_\gamma,\alpha_{\gamma'}}{}^l(q^{-1}) &= \frac{1}{2\pi}(\cosh\tfrac{1}{2}q)^{-2-2l} \int_1^\infty dx\, x^{l+\alpha_\gamma} \\ &\quad\times (x+\tanh\tfrac{1}{2}q)^{-l-1-\alpha_{\gamma'}} \\ &\quad\times (1+x\tanh\tfrac{1}{2}q)^{-l-1+\alpha_{\gamma'}}\,, \\ R^{\gamma\gamma'}(k,w;k',w') &\equiv \tilde{\beta}^{\gamma_l\gamma_l'}(t_l,t_l')[\beta^{\gamma_u\gamma_u'}(t_u,t_u')]^* \\ &\quad\times (\sinh q)^{\alpha_\gamma+\alpha_{\gamma'}} k^{\alpha_{\gamma'}} k'^{\alpha_\gamma}\,, \end{aligned} \tag{4.20}$$

and α_γ and q have been defined in Eqs. (3.7) and (3.10). Note that the expression given above for $R^{\gamma\gamma'}$ is valid only for the leading term in the sequence of $O(1,1)$ poles corresponding to a single Regge pole in the BCP expansion. Though our crossed partial-wave analysis is general, these leading terms are the only ones explicitly accounted for in this paper.

[29] The expression for ${}_{(0)}b^l$ is the one given below if $z_0>0$. If $z_0<0$, $Q_l{}^{\alpha_\gamma}$ has to be replaced by a more complicated expression derived from (A25).

The most singular part in the l plane of the kernel of Eq. (4.19) is given by

$$\begin{aligned} (l+1-\alpha_\gamma)^{-1} 2^l \tilde{\beta}^{\gamma_l\gamma_l'}(\tilde{\beta}^{\gamma_u\gamma_u'})^*[(\tanh\tfrac{1}{2}q)^{\alpha_{\gamma'}} + (\tanh\tfrac{1}{2}q)^{-\alpha_{\gamma'}}] \\ \times(\sinh q)^{-l-1+\alpha_\gamma+\alpha_{\gamma'}} k^{\alpha_{\gamma'}} k'^{\alpha_\gamma}\,. \end{aligned} \tag{4.21}$$

It is interesting to compare it with the kernels obtained by using the Mellin-transform technique with an asymptotic representation of the phase space.[5,7] One striking difference is the presence of the last three factors. For small k this term factorizes in k and k' and, after a redefinition of $b_\gamma{}^l$, yields a "threshold" factor $(k^2)^{l-l_c}$, where $l_c=\alpha_{\gamma_l}(t_l)+\alpha_{\gamma_u}(t_u)-1$. This factor can be neglected when l is close to the branch point, where the output Regge pole occurs in weak-coupling models. This is also the limit in which the Mellin-transform approach is most plausible.

An additional feature of our kernel is the presence, through a dependence on $\sinh q$, of a kinematical correlation between the k and k' variables for $k, k' \gtrsim m$, where m is the mass of the outgoing particle(s) at the vertex. For linear input trajectories this also provides a natural cutoff at large values of k.

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APPENDIX A

$O(2,1)$ in a Noncompact Basis

We summarize here the properties of the representation functions of the $O(2,1)$ group in noncompact bases which are relevant to our paper. The reason is that we use a slightly different basis than Mukunda,[22] and also that the representation functions in the $O(2)\times O(1,1)$ basis are not found in the literature.

We are interested in the matrix elements of transformations like

$$e^{-iJ_z\phi}e^{-iK_x\eta}e^{-iK_y\xi}\,, \tag{A1}$$

which connect timelike to spacelike three-momenta and transformations of the form

$$e^{-iK_y\xi'}e^{-iK_x\eta}e^{-iK_y\xi''} \tag{A2}$$

for the spacelike-spacelike case. Although the latter parametrization of the $O(2,1)$ group is not complete, it is sufficient for our purposes, due to the form (2.22) of a_i. We shall use the mixed basis for transformations (A1) and the $O(1,1)$ basis for (A2), with the definition

$$d_{m,\mu r}{}^l(\eta)=\langle l,m|e^{-iK_x\eta}|l,\rho r\rangle=[d_{\mu r,m}{}^l(\eta^{-1})]^*,$$
$$d_{\mu r,\mu' r'}{}^l(\eta)=\langle l,\rho r|e^{-iK_x\eta}|l,\rho' r'\rangle=[d_{\mu' r',\mu r}{}^l(\eta^{-1})]^*, \tag{A3}$$

where

$$\mu=i\rho,\quad \mu'=i\rho'.$$

The representation of $O(2,1)$ suitable for the $O(1,1)$ basis is defined[22,23] in the Hilbert space of the functions $f_s(\xi)$ $(s=1,2)$ with the scalar product

$$(f,g)=\sum_s\int_{-\infty}^{+\infty}d\xi\, f_s{}^*(\xi)g_s(\xi). \tag{A4}$$

In this Hilbert space we shall choose, for the $|l,\rho\pm\rangle$ states, the particular representations given, respectively, by

$$\frac{1}{(2\pi)^{1/2}}\begin{pmatrix}1\\0\end{pmatrix}e^{i\rho\xi}\quad\text{and}\quad\frac{1}{(2\pi)^{1/2}}\begin{pmatrix}0\\1\end{pmatrix}e^{i\rho\xi}. \tag{A5}$$

This choice is different from Mukunda's.[22] The x boost is represented, in this Hilbert space, by

$$e^{+iK_x\eta}f_s(\xi)=f_s'(\xi)\quad(\eta>0), \tag{A6}$$

where

$$f_1'(\xi)=(\cosh\eta+\cosh\xi\sinh\eta)^{-l-1}f_1(\xi'),$$
$$\begin{aligned}f_2'(\xi)={}&(\cosh\xi\sinh\eta-\cosh\eta)^{-l-1}f_1(\xi_1)\\&\times\theta(\cosh\xi\sinh\eta-\cosh\eta)\\&+(\cosh\eta-\cosh\xi\sinh\eta)^{-l-1}f_2(\xi_2)\\&\times\theta(\cosh\eta-\cosh\xi\sinh\eta),\end{aligned} \tag{A7}$$

$$e^{\xi'}=\frac{e^\xi+\tanh\frac12\eta}{1+e^\xi\tanh\frac12\eta},\quad e^{\xi_1}=\frac{e^\xi-\tanh\frac12\eta}{e^\xi\tanh\eta-1},$$
$$e^{\xi_2}=\frac{e^\xi-\tanh\frac12\eta}{1-e^\xi\tanh\frac12\eta}.$$

$O(1,1)$ Basis

By substituting (A5)–(A7) into the second definition (A3), we get, for example,[30]

$$d_{\mu'+,\mu+}{}^l(\eta^{-1})=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\xi\, e^{-\mu'\xi}(\cosh\eta+\cosh\xi\sinh\eta)^{-l-1}\times\left(\frac{e^\xi+\tanh\frac12\eta}{1+e^\xi\tanh\frac12\eta}\right)^\mu. \tag{A8}$$

[30] We prefer to give directly the representation functions occurring in the kernel (4.7) and in the amplitudes (B1) instead of those in the expansion (4.1). They are related by complex conjugation.

By using the relation

$$\cosh\eta+\cosh\xi\sinh\eta=(e^\xi+\tanh\tfrac12\eta)(e^{-\xi}+\tanh\tfrac12\eta)\times(\cosh\tfrac12\eta)^2, \tag{A9}$$

we can reduce (A8) to a standard representation of a hypergoemetric function,[31] and we obtain

$$\begin{aligned}d_{\mu'+,\mu+}{}^l(\eta^{-1})={}&\frac{1}{2\pi}(\sinh\tfrac12\eta)^{-2l-2}(\tanh\tfrac12\eta)^{-\mu-\mu'}\\&\times\frac{\Gamma(l+1+\mu')\Gamma(l+1-\mu')}{\Gamma(2l+2)}\\&\times F(l+1+\mu,\,l+1+\mu';\,2l+2;\,-(\sinh\tfrac12\eta)^{-2})\\={}&d_{(-\mu')+,(-\mu)+}{}^l(\eta^{-1}).\end{aligned} \tag{A10}$$

In the same way we have

$$d_{\mu'-,\mu-}{}^l(\eta^{-1})=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\xi\, e^{-\mu'\xi}\frac{\theta(\cosh\eta-\cosh\xi\sinh\eta)}{(\cosh\eta-\cosh\xi\sinh\eta)^{l+1}}\times\left(\frac{e^\xi-\tanh\frac12\eta}{1-e^\xi\tanh\frac12\eta}\right)^\mu, \tag{A11}$$

and by changing variables to

$$e^{\xi'}=(e^\xi-\tanh\tfrac12\eta)/(1-e^\xi\tanh\tfrac12\eta), \tag{A12}$$

we get the result

$$d_{\mu'-,\mu-}{}^l(\eta^{-1})=d_{\mu+,\mu'+}{}^{-l-1}(\eta^{-1}). \tag{A13}$$

As for the plus-minus matrix elements, we discover that

$$d_{\mu'+,\mu-}{}^l(\eta^{-1})=d_{\mu-,\mu'+}{}^l(\eta)=0\quad\text{for }\eta>0, \tag{A14}$$

which shows the convenience of our basis (A5). The last matrix element is not zero and is

$$\begin{aligned}d_{\mu'-,\mu+}{}^l(\eta^{-1})={}&\frac{1}{2\pi}\left(\int_{-\infty}^{-\xi_0}+\int_{\xi_0}^{\infty}\right)d\xi\\&\times e^{-\mu'\xi}(\cosh\xi\sinh\eta-\cosh\eta)^{-l-1}\left(\frac{e^\xi-\tanh\frac12\eta}{e^\xi\tanh\frac12\eta-1}\right)^\mu\\&(e^{-\xi_0}=\tanh\tfrac12\eta).\end{aligned} \tag{A15}$$

After a change of variables similar to (A12) we get

$$\begin{aligned}&d_{\mu'-,\mu+}{}^l(\eta^{-1})\\&=d_{\mu-,\mu'+}{}^{-l-1}(\eta^{-1})=\frac{1}{2\pi}(\cosh\tfrac12\eta)^{-2l-2}(\tanh\tfrac12\eta)^{\mu-\mu'}\\&\times\frac{\Gamma(l+1-\mu')\Gamma(\mu-l)}{\Gamma(1+\mu-\mu')}\\&\times F(l+1+\mu,\,l+1-\mu';\,\mu-\mu'+1;\,(\tanh\tfrac12\eta)^2)\\&+(\mu\leftrightarrow-\mu;\,\mu'\leftrightarrow-\mu').\end{aligned} \tag{A16}$$

[31] *Bateman Manuscript Project*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953), Vol. I, Sec. 2.12.

Equations (A10), (A13), (A14), and (A16) give the desired results. In particular,[32]

$$d_{\mu+,0+}{}^l(\eta^{-1})=\frac{1}{\pi}\frac{\Gamma(l+1-\mu)}{\Gamma(l+1)}Q_l{}^\mu(\cosh\eta)\,,$$
$$d_{0+,\mu+}{}^l(\eta^{-1})=\frac{1}{\pi}\frac{\Gamma(l+1)}{\Gamma(l+1+\mu)}Q_l{}^\mu(\cosh\eta)\,. \tag{A17}$$

We finally mention, without proof, the relation

$$\cos\pi l\, d_{\mu'-,\mu+}{}^l(\eta^{-1})=-\cos\pi\mu'\, d_{\mu'+,\mu+}{}^l(\eta^{-1})+\cos\pi\mu\, d_{\mu'-,\mu-}{}^l(\eta^{-1})\,, \tag{A18}$$

valid when $\eta>0$. It can be used to prove that Eq. (4.9b) of the text is actually solved by relation (B1) between $(-)$ and $(+)$ amplitudes derived below.

Mixed Basis

We can obtain the representation functions in the $O(2)\times O(1,1)$ basis by using the same Hilbert space as before, by using the representation of the states $|lm\rangle$ in this space,[22,23] which is

$$\begin{pmatrix} f_m(\xi)\\ f_{-m}(\xi)\end{pmatrix},\quad f_m(\xi)=\frac{1}{(2\pi)^{1/2}}(\cosh\xi)^{-l-1}\left(\frac{1+ie^\xi}{1-ie^\xi}\right)^m. \tag{A19}$$

From (A7), after some algebra, we obtain ($\eta>0$)

$$d_{\mu\pm,m}{}^l(\eta^{-1})=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\xi\times e^{-\mu\xi}(\cosh\xi\cosh\eta\pm\sinh\eta)^{-l-1}e^{im\phi\pm(\xi)}\,, \tag{A20}$$

where

$$\tan\tfrac{1}{2}\phi_\pm=(e^\xi\pm\tanh\tfrac{1}{2}\eta)/(e^\xi\tanh\tfrac{1}{2}\eta\pm1)\,. \tag{A21}$$

For $m=0$, $r=+$, (A20) is a standard representation of a Q_l function,[33] and we obtain, for $\eta>0$,

$$d_{\mu+,0}{}^l(\eta^{-1})=\frac{1}{\pi}\frac{\Gamma(l+1-\mu)}{\Gamma(l+1)}i^{l+1}Q_l{}^\mu(i\sinh\eta)=d_{\mu-,0}{}^l(\eta)\,, \tag{A22}$$

where the last equation follows from the relation[22]

$$e^{i\pi J_z}|\rho+\rangle=|-\rho,-\rangle\,, \tag{A23}$$

and from the fact that the $d_{\mu+,0}{}^l$ is even in μ.

For $m=0$, $r=-$, the right-hand side of Eq. (A20) is proportional to the analytic continuation of $Q_l{}^\mu(i\sinh\eta)$ from $\eta>0$ to $\eta<0$ onto the Riemann sheet reached through the cut $-1<z<1$. Therefore, by making use of the discontinuity formula[34]

$$Q_l{}^\mu(x+i0)=e^{i\mu\pi}Q_l{}^\mu(x-i0)-i\pi P_l{}^\mu(x-i0)\,, \tag{A24}$$

we get

$$d_{\mu-,0}{}^l(\eta^{-1})=d_{\mu+,0}{}^l(\eta)=-\frac{\cos\pi\mu}{\cos\pi l}d_{\mu+,0}{}^l(\eta^{-1})+\frac{\pi\Gamma(-l)}{\Gamma(l+1)\cos\pi l}\frac{d_{\mu+,0}{}^{-l-1}(\eta^{-1})}{\Gamma(\mu-l)\Gamma(-\mu-l)}\,. \tag{A25}$$

Group Properties of Q_l Functions

By the use of the r index it is possible to have pure Q_l-type representation functions, thus providing group-theoretical properties for $Q_l{}^\mu(z)$. For instance, using (A14), we obtain

$$\langle l,0+|e^{iK_x\eta_1}e^{iK_y\xi}e^{iK_x\eta_2}|l,0+\rangle=(-i)\int_{-i\infty}^{+i\infty}d\mu\, d_{0+,\mu+}{}^l(\eta_1{}^{-1})e^{\mu\xi}d_{\mu+,0+}{}^l(\eta_2{}^{-1})\,, \tag{A26}$$

and using (A17), we get the addition theorem (for $z_i=\cosh\eta_i$)

$$Q_l[z_1z_2+(z_1{}^2-1)^{1/2}(z_2{}^2-1)^{1/2}\cosh\xi]=(-i)\int_{-i\infty}^{+i\infty}\frac{d\mu}{2\pi}Q_l{}^\mu(z_1)Q_l{}^{-\mu}(z_2)e^{-\mu\xi}\,. \tag{A27}$$

When $e^\xi>\coth\frac{1}{2}\eta_1\coth\frac{1}{2}\eta_2$, or if $z_i=i\sinh\eta_i$, the μ contour can be closed in $\mathrm{Re}\mu>0$, picking up the poles of $Q_l{}^{-\mu}$ at $\mu=l+1+n$ and giving (A27) a form known in the literature.[35]

The result (A27) can be used to give the crossed partial-wave analysis of the AFS-type equation (3.16) without explicit use of the group theory.[36]

APPENDIX B: RELATION BETWEEN + AND − AMPLITUDES

We have seen in the text that the $(+)$ amplitude can be determined separately from Eq. (4.9a). Then Eq. (4.9b) gives $B_-{}^l$ in terms of $B_+{}^l$.

Note first that the only additional Regge poles which can arise from (4.9b) come from the singular points of $(1-K_{--}{}^l)^{-1}$ and, since $K_{--}{}^l$ is related to $K_{++}{}^{-l-1}$ [Eq. (A13)], they are simply the Regge poles at the symmetric points $l'=-l-1$. (Remember that the output amplitude A_l is symmetric under $l\leftrightarrow -l-1$.) Therefore, only $K_{++}{}^l$ is relevant for determining the position of the output Regge poles.

On the other hand, an explicit simple relation between $(-)$ and $(+)$ amplitudes can be found if $\eta>0$ in the

[32] In our $Q_l{}^\mu$ functions the factor $e^{i\mu\pi}$ of the definition in Ref. 31 has been dropped throughout.

[33] Reference 31, Sec. 3.7.

[34] Reference 31, Sec. 3.4.

[35] F. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics* (Cambridge University Press, New York, 1931), p. 384.

[36] M. Ciafaloni, University of California, Berkeley (unpublished).

parametrization (3.17) of a. In such a case, from the definition (4.1) of $B_{\mu r}{}^l$ and from relation (A25), we get

$$B_{\mu+}{}^l=\int_{-\infty}^{+\infty}d\xi\, e^{\mu\xi}\int_{\eta_{\min}}^{\infty}d\sinh\eta\; d_{\mu+,0}{}^l(\eta^{-1})B(\eta,\xi)\,,$$

$$B_{\mu-}{}^l=-\frac{\cos\pi\mu}{\cos\pi l}B_{\mu+}{}^l+\frac{\pi\Gamma(-l)}{\Gamma(l+1)\cos\pi l}\frac{B_{\mu+}{}^{-l-1}}{\Gamma(\mu-l)\Gamma(-\mu-l)}\,, \qquad \text{(B1)}$$

which solves explicitly, for this case, the system of equations (4.9). This can be verified in a straightforward way by using (A18) given above to relate $K_{+-}{}^l$ to $K_{++}{}^l$ and $K_{++}{}^{-l-1}$.

Note that $\sinh\eta$, given by (3.18), can be negative when the energy $s<E_a{}^2=m_a{}^2-\frac{1}{4}t$. This can occur, however, only for the first few links for any fixed t. The foregoing argument, therefore, strictly holds for that part of the absorptive part which comes from intermediate states of sufficiently high multiplicity.[20] In some multiperipheral models of, e.g., π-π and π-N scattering, when t is in the region of the forward peak, the first average subenergy is already large enough to make the case $\eta<0$ (and the occurrence of timelike **k**'s) presumably unimportant from the second link on. In such cases the procedure of Ref. 20 involves only the separate treatment of the elastic unitarity graph $A_2(a)$.

APPENDIX C: GENERALIZATION TO TOLLER-ANGLE DEPENDENCE

We indicate here, for the sake of completeness, how our equations are modified in the case of Toller-angle dependence of the production amplitudes. We use a method of Mueller and Muzinich,[37] which essentially consists in adding an extra index $\tau=\operatorname{sgn}\zeta$ to the incomplete absorptive part.

As remarked in Eq. (3.5), the Toller angle ω_i depends on τ_i and τ_{i+1}. This means that the residues at the poles $\mu=\alpha$ and $\mu=-\alpha$ are different. Therefore, we must treat positive and negative ζ's separately.

[37] A. H. Mueller and I. J. Muzinich, Brookhaven National Laboratory Report No. BNL-13836, 1969 (unpublished).

The $O(1,1)$ expansion of the production amplitudes becomes

$$M^{(n+2)}\sim\sum_{m_a,m_b,\gamma_i,\tau_i}e^{im_a\phi_a}\beta_{m_a\tau_1}{}^{\gamma_1}e^{\alpha_{\gamma_1}\tau_1\zeta_1}\theta(\tau_1\zeta_1)\beta_{\tau_1\tau_2}{}^{\gamma_1\gamma_2}\times e^{\alpha_{\gamma_2}\tau_2\zeta_2}\theta(\tau_2\zeta_2)\cdots\beta_{\tau_{n+1}m_b}{}^{\gamma_{n+1}}e^{im_b\phi_b}\,, \qquad \text{(C1)}$$

where the k,w variables have been dropped. For the incomplete absorptive part we now have the equation

$$B_{\tau'}{}^{\gamma'}(a';k',w';t)={}_{(0)}B_{\tau'}{}^{\gamma'}(a';k',w';t)+\tfrac{1}{2}\sum_{\gamma,\tau}\int dkdwd\zeta' B_{\tau}{}^{\gamma}(a'\zeta'^{-1}q^{-1};t)\times R_{\tau\tau'}{}^{\gamma\gamma'}(k,w;k',w';t)e^{\alpha_{\gamma'}(k',w',t)\tau'\zeta'}\theta(\tau'\zeta')\,, \qquad \text{(C2)}$$

where

$${}_{(0)}B_{\tau'}{}^{\gamma'}\equiv\sum_{m_{la},m_{ua}}\frac{1}{2E_a}\delta(\sinh q_0-z_0)e^{im_a\phi_a}\times R_{m_{la},m_{ua},\tau'}{}^{\gamma'}e^{\alpha_{\gamma'}\tau'\zeta'}\theta(\tau'\zeta')\,, \qquad \text{(C3)}$$

and $R_{\tau\tau'}{}^{\gamma\gamma'}$ is defined as in Eq. (4.20), but with the Toller-angle dependence included in $\tilde{\beta}^{\gamma\gamma'}$.

In the diagonalization, the amplitude $B_{\mu r\tau}{}^l$ has only the pole $\mu=-\tau\alpha$, and the partially diagonalized Eq. (4.6) is replaced by

$$B_{\mu'r'\tau'}{}^l={}_{(0)}B_{\mu'r'\tau'}{}^l+\sum_{r\tau}\int_{-i\infty}^{+i\infty}(-i)d\mu\, B_{\mu r\tau}{}^l\times d_{\mu'r',\mu r}{}^l(q^{-1})R_{\tau\tau'}(-\tau')/(\mu'+\tau'\alpha')\,. \qquad \text{(C4)}$$

The separation of right-hand and left-hand kinematical singularities in the μ plane for $d_{\mu'+,\mu+}{}^l$ proceeds as before except that, for a given τ, only one of the functions $\hat{d}_{\mu,\mu'}{}^l$ and $\hat{d}_{-\mu,-\mu'}{}^l$ contributes. The final equation is

$$b_{\gamma'\tau'}{}^l(k',w')={}_{(0)}b_{\gamma'\tau'}{}^l(k',w')+\pi\sum_{\gamma,\tau}\int dkdw\, b_{\gamma\tau}{}^l(k,w)\times R_{\tau\tau'}{}^{\gamma\gamma'}(k,w;k',w';t)\hat{d}_{\alpha_\gamma,\tau\tau'\alpha_{\gamma'}}{}^l(q^{-1})\,, \qquad \text{(C5)}$$

where ${}_{(0)}b_{\gamma\tau}{}^l$ is defined as in Eq. (4.20), except that now, e.g., $\tilde{\beta}^{\gamma l}=\tilde{\beta}^{\gamma l}(t_l,\tau)$.

Equation (4.19) follows from (C5) in the case of τ independence of the residue functions.